

C.1 Simple harmonic motion

Ahmed Gomaa · IB DP Physics SL and HL (sections marked HL are for HL only) · uses A.2 (Hooke's law, $F = ma$) and circular motion (ω) · original worked examples

Contents

- | | |
|--|--|
| 1. <u>The big picture</u> | 10. <u>HL · The equations, by differentiating.</u> |
| 2. <u>Words that describe an oscillation</u> | |
| 3. <u>The conditions for SHM</u> | 11. <u>HL · Speed at any displacement</u> |
| 4. <u>From Hooke's law to $a = -\omega^2 x$</u> | 12. <u>HL · Phase angle: the data booklet equation</u> |
| 5. <u>The simple pendulum</u> | |
| 6. <u>Any system: why we use ω^2</u> | 13. <u>HL · Energy equations</u> |
| 7. <u>The acceleration-displacement graph</u> | 14. <u>HL · The vertical spring.</u> |
| 8. <u>Displacement, velocity and acceleration graphs</u> | 15. <u>Writing the explanations</u> |
| 9. <u>Energy in SHM</u> | 16. <u>Common mistakes</u> |
| | 17. <u>Practise next</u> |

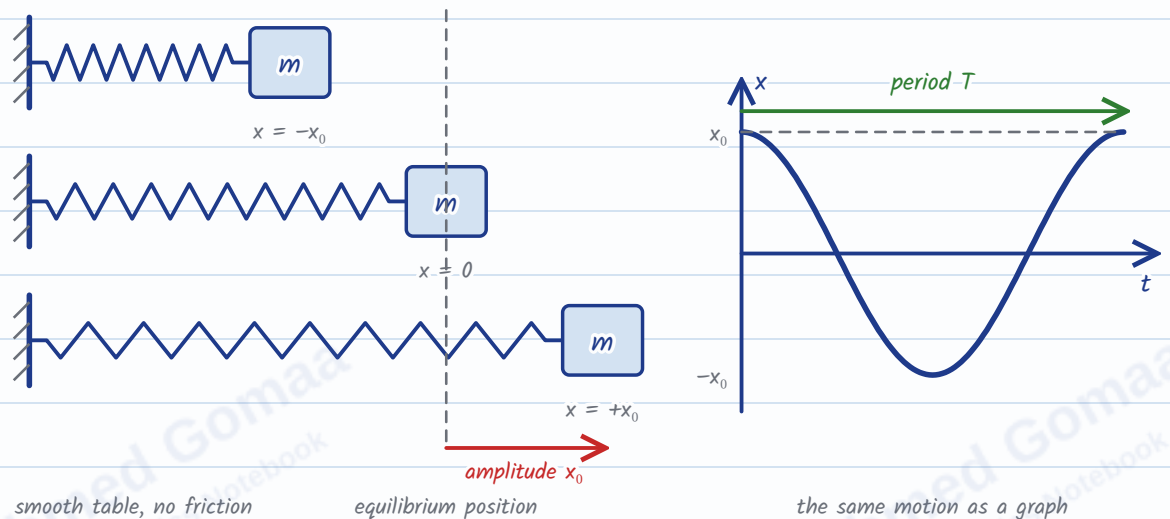
Pen colours: blue = objects and displacement, red = forces, velocity and key steps, green = acceleration, answers and results, pencil = equilibrium and construction lines. Boxes: **Why?** = the reasoning, **Mistake** = what students get wrong, yellow note = exam tip, dashed box = a derivation step by step.

The big picture

An **oscillation** is a motion that repeats, back and forth about an **equilibrium position**: a mass on a spring, a pendulum, a guitar string, an atom in a solid, a skyscraper in the wind. Simple harmonic motion (SHM) is the simplest kind, and it is the model for all the others. It is also the start

of Theme C: when a wave passes, every particle of the medium moves in SHM.

You already have the two tools you need. From A.2: Hooke's law $F = -kx$ and Newton's second law $F = ma$. From circular motion: the angular speed $\omega = 2\pi/T$. Put them together and you get the one equation of this topic, $a = -\omega^2 x$.



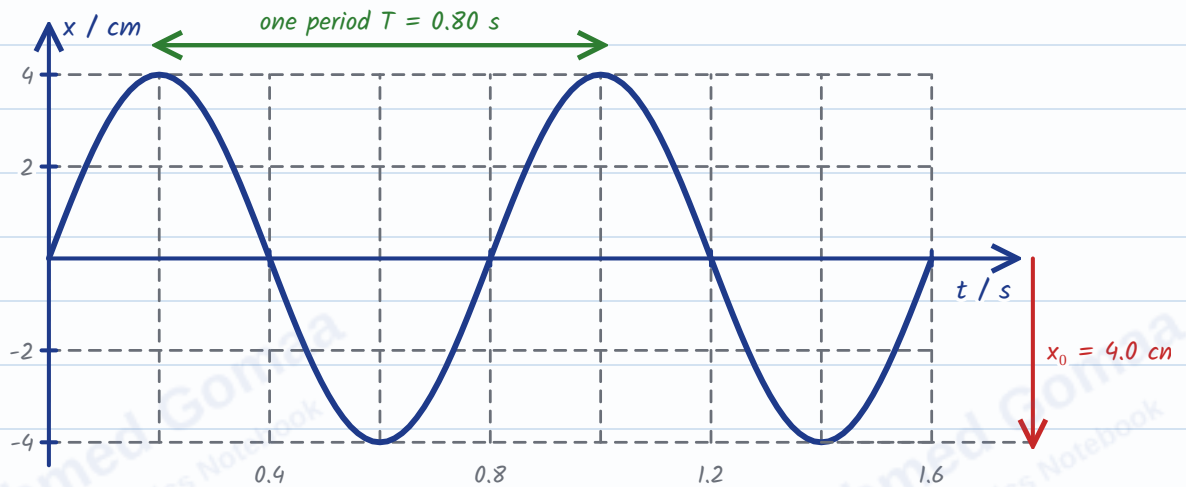
A block on a spring at its two ends and in the middle. The displacement-time graph of the same motion is a sine or cosine curve.

Words that describe an oscillation

Word	Symbol and unit	Meaning
equilibrium position	$x = 0$	the position where the net force on the object is zero
displacement	x, m	the distance from the equilibrium position, in a stated direction (it can be negative)
amplitude	x_0, m	the maximum displacement from the equilibrium position
period	T, s	the time taken for one complete oscillation

frequency	f , Hz	the number of oscillations per unit time
angular frequency	ω , rad s^{-1}	$\omega = 2\pi f$: the frequency written as an angle per second (one oscillation = 2π rad)

$$T = 1/f = 2\pi/\omega \cdot \omega = 2\pi f = 2\pi/T$$



Read the amplitude from the time axis to a peak (4.0 cm), not from a peak to a trough (8.0 cm). Read the period from one peak to the next.

Worked example 1

A particle completes 5 full oscillations in 4.0 s. Find the period, the frequency and the angular frequency.

1. $T = 4.0/5 = 0.80 \text{ s}$.
2. $f = 1/T = 1/0.80 = 1.25 \text{ Hz}$.
3. $\omega = 2\pi/T = 2\pi/0.80 = 7.85 \text{ rad s}^{-1}$.

✗ Mistake

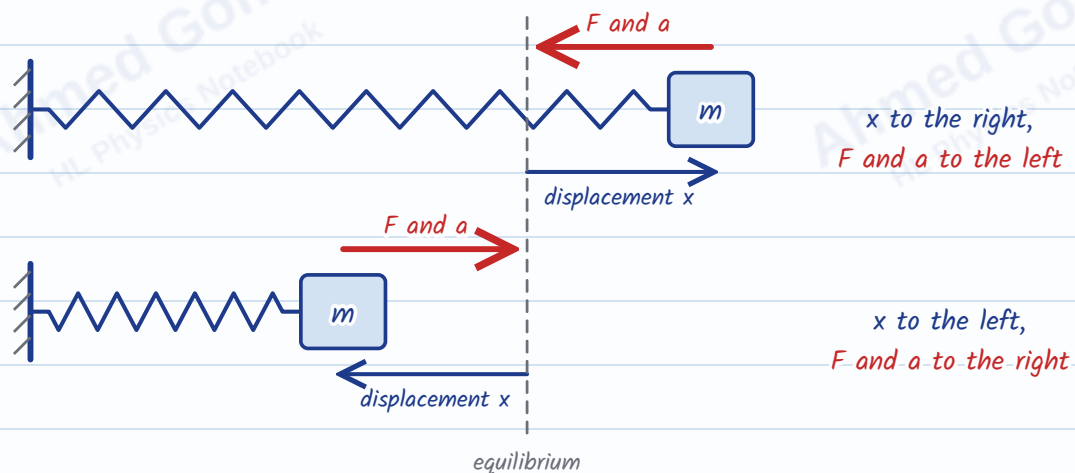
Taking the amplitude as the distance from one end to the other. That distance is $2x_0$. A ruler that moves 12 cm from end to end has an amplitude of 6 cm.

The conditions for SHM

Any oscillation needs a **restoring force**: a force that pulls the object back towards the equilibrium position. SHM is a special case: the restoring force grows in exact proportion to the displacement.

Conditions for simple harmonic motion (2 marks) · learn it word for word

- ✓ the acceleration (or the restoring force) is **proportional to the displacement** from the equilibrium position
- ✓ and it is **directed towards the equilibrium position** (opposite to the displacement).



Displace the block either way: the force, and so the acceleration, always points back to equilibrium. Double the displacement and the force doubles.

Defining equation of SHM: $a = -\omega^2 x$

$a \propto x$: "proportional to the displacement" · the minus sign: "opposite to the displacement"

The meaning of the minus sign

ω^2 is always positive. So the minus sign makes a and x always have opposite signs: when x is positive, a is negative, and the other way round. In words: the acceleration always points towards the equilibrium position. Without the minus sign the object would accelerate away and never come back.

✗ Mistake

Answering "state the conditions for SHM" with the equation only. The markscheme says **do not award marks for stating the defining equation**: $a = -\omega^2 x$ on its own scores 0, and at most 1 mark even if you define a and x . Write the two conditions in words.

Two more facts about SHM

- ✓ The period does **not depend on the amplitude** (the motion is isochronous): a bigger swing takes the same time.
- ✓ With no friction, the amplitude stays constant. If the period changes, or the amplitude changes, the motion is not SHM.

From Hooke's law to $a = -\omega^2 x$

This is where $a = -\omega^2 x$ comes from. Take a block of mass m on a spring of spring constant k , on a smooth table.

Build it step by step

1. Displace the block by x . The spring pulls back with Hooke's law: $F = -kx$ (the minus sign: the force is opposite to x).
2. This is the only horizontal force, so it is the net force. Newton's second law: $ma = -kx$.
3. Divide by m : $a = -(k/m) x$.

4. Compare with the defining equation $a = -\omega^2 x$. The bracket is ω^2 : $\omega^2 = k/m$.

5. So $\omega = \sqrt{k/m}$, and $T = 2\pi/\omega$ gives $T = 2\pi\sqrt{m/k}$.

Mass-spring system: $T = 2\pi\sqrt{m/k}$ (data booklet)

m = the oscillating mass (kg) · k = spring constant (N m^{-1})

Important note

$\omega^2 = k/m$ is true **only for a mass-spring system**. Do not use k/m for a pendulum, a floating block, a molecule or any other system. For any other system, find what plays the part of k/m (see the next two sections), or use ω^2 straight from the data: $\omega = 2\pi/T$, or from the gradient of an a - x graph.

Worked example 2

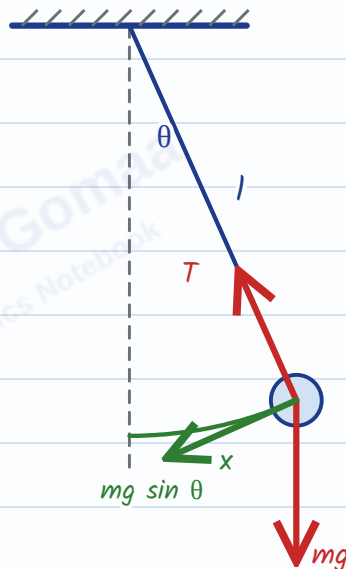
A mass on a spring of spring constant 180 N m^{-1} oscillates with a period of 0.90 s . (a) Find the mass. (b) The mass is halved and the spring constant is doubled. What happens to the period?

1. (a) $T = 2\pi\sqrt{m/k}$, so $m = kT^2/(4\pi^2) = 180 \times 0.90^2/(4\pi^2) = 3.69 \text{ kg}$.

2. (b) $T \propto \sqrt{m/k}$. m/k becomes $(m/2)/(2k) = (m/k)/4$, so T becomes $\sqrt{1/4} = 1/2$ of its value: **the period halves**.

The simple pendulum

A simple pendulum is a small heavy bob on a light string of length l . Pull it aside by a small angle and let go.



$$\text{restoring force} = mg \sin \theta$$

$$\text{small } \theta \text{ (radians): } \sin \theta \approx \theta = x/l$$

$$a = -g \sin \theta \approx -(g/l) x$$

$$\text{compare with } a = -\omega^2 x:$$

$$\omega^2 = g/l$$

$$T = 2\pi\sqrt{l/g}$$

The restoring force is the part of the weight along the arc, $mg \sin \theta$. For small angles it is proportional to x : SHM.

Same method as the spring

1. Restoring force along the arc: $F = -mg \sin \theta$ (towards equilibrium).
2. For a small angle in radians, $\sin \theta \approx \theta$, and $\theta = x/l$. So $F \approx -mg x/l$.
3. Newton's second law: $ma = -(mg/l) x$, so $a = -(g/l) x$.
4. Compare with $a = -\omega^2 x$: $\omega^2 = g/l$, and $T = 2\pi\sqrt{l/g}$.

Simple pendulum: $T = 2\pi\sqrt{l/g}$ (data booklet) · only for **small** angles (below about 10°)

the period does **not** depend on the mass of the bob

Worked example 3

Find the length of a pendulum with a period of 2.0 s on Earth ($g = 9.8 \text{ m s}^{-2}$). If the length is made four times larger, what is the new period?

$$1. l = gT^2/(4\pi^2) = 9.8 \times 2.0^2/(4\pi^2) = 0.99 \text{ m.}$$

$$2. T \propto \sqrt{l}, \text{ so 4 times the length gives } \sqrt{4} = 2 \text{ times the period: } 4.0 \text{ s.}$$

✗ Mistake

Thinking a heavier bob swings faster or slower. The mass cancels (it is in both the force mg and in ma), so the period of a pendulum does not depend on the mass.

Any system: why we use ω^2

For a spring, $\omega^2 = k/m$. For a pendulum, $\omega^2 = g/l$. Every system that oscillates in SHM gives its own expression, but all of them end in the same shape, $a = -(\text{something}) x$. So the IB writes the general equation with one symbol, ω^2 , that works for every system.

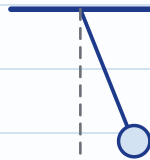


spring

$$F = -kx$$

$$a = -(k/m) x$$

$$\omega^2 = k/m$$



pendulum (small θ)

$$F \approx -(mg/l) x$$

$$a = -(g/l) x$$

$$\omega^2 = g/l$$



two springs

$$F = -kx - kx = -2kx$$

$$a = -(2k/m) x$$

$$\omega^2 = 2k/m$$

Same method every time: find the restoring force, write $a = -(\dots) x$, and the bracket is ω^2 .

One method for every system. Whatever is in the bracket is ω^2 .

Important note

Use $a = -\omega^2 x$ for any system in SHM. Use k/m only for a mass-spring system, and g/l only for a pendulum. When the question gives you a graph or a period, find ω from the data ($\omega = 2\pi/T$, or $\omega^2 = -\text{gradient of the } a-x \text{ graph}$), not from k/m .

Worked example 4

A 1.5 kg block on a smooth table is connected to two springs, one on each side, each of spring constant 150 N m^{-1} . Find the period of the oscillation.

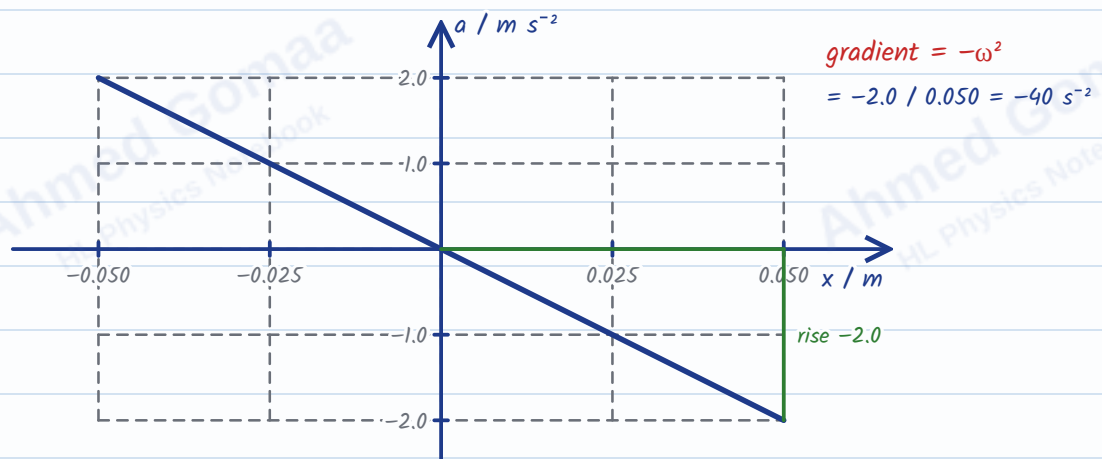
1. Displace the block by x : one spring is stretched by x and the other is squashed by x . Both push the block back: $F = -kx - kx = -2kx$.
2. $a = -(2k/m)x$, so $\omega^2 = 2k/m = 2 \times 150/1.5 = 200 \text{ s}^{-2}$, $\omega = 14.1 \text{ rad s}^{-1}$.
3. $T = 2\pi/\omega = 2\pi/14.1 = 0.44 \text{ s}$.

The acceleration-displacement graph

$a = -\omega^2 x$ has the same shape as $y = mx$ with m negative. So for SHM, the graph of acceleration against displacement is a straight line through the origin with a negative gradient, and

gradient of the a - x graph $= -\omega^2$

so $\omega = \sqrt{(-\text{gradient})}$, and then $T = 2\pi/\omega$, $f = \omega/2\pi$



The line ends at $\pm x_0$; the object never goes further than the amplitude. The largest acceleration, 2.0 m s^{-2} , is at the ends.

How to explain from the graph that the motion is SHM (2 marks)

- ✓ Straight line through the origin, so a is proportional to x .
- ✓ Negative gradient, so a is opposite to x (directed towards equilibrium).

Two features of the graph, two conditions, two marks.

Worked example 5

The graph above shows the variation of the acceleration a with the displacement x of an object of mass 0.25 kg . (a) Explain why the motion is simple harmonic. (b) Find the period. (c) Find the maximum speed. (d) Find the force on the object when $x = -0.030 \text{ m}$.

1. (a) The line passes through the origin, so $a \propto x$; the gradient is negative, so a is opposite to x : **SHM**.
2. (b) Gradient $= -2.0/0.050 = -40 \text{ s}^{-2}$. So $\omega^2 = 40$, $\omega = \sqrt{40} = 6.32 \text{ rad s}^{-1}$, and $T = 2\pi/6.32 = \mathbf{0.99 \text{ s}}$.
3. (c) The amplitude is the end of the line: $x_0 = 0.050 \text{ m}$. $v_{\max} = \omega x_0 = 6.32 \times 0.050 = \mathbf{0.32 \text{ m s}^{-1}}$.
4. (d) $F = ma = m(-\omega^2 x) = 0.25 \times (-40) \times (-0.030) = \mathbf{+0.30 \text{ N}}$ (positive: towards equilibrium).

Worked example 6

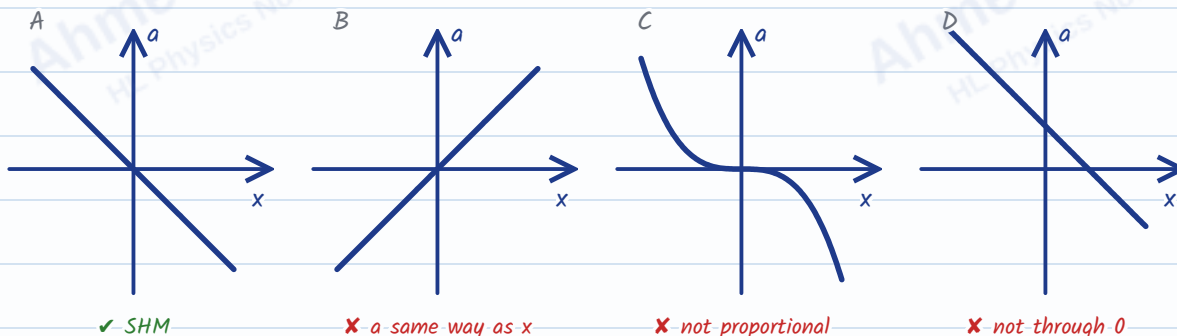
For another object in SHM, the a - x graph is a straight line through the origin that passes through the point ($x = -0.10 \text{ m}$, $a = 3.0 \text{ m s}^{-2}$).

Find f^2 and then f .

1. Gradient $= 3.0/(-0.10) = -30 \text{ s}^{-2}$, so $\omega^2 = 30 \text{ s}^{-2}$.
2. $\omega = 2\pi f$, so $\omega^2 = 4\pi^2 f^2$ and $f^2 = \omega^2/(4\pi^2) = 30/(4\pi^2) = \mathbf{0.76 \text{ s}^{-2}}$.
3. $f = \sqrt{0.76} = \mathbf{0.87 \text{ Hz}}$.

✗ Mistake

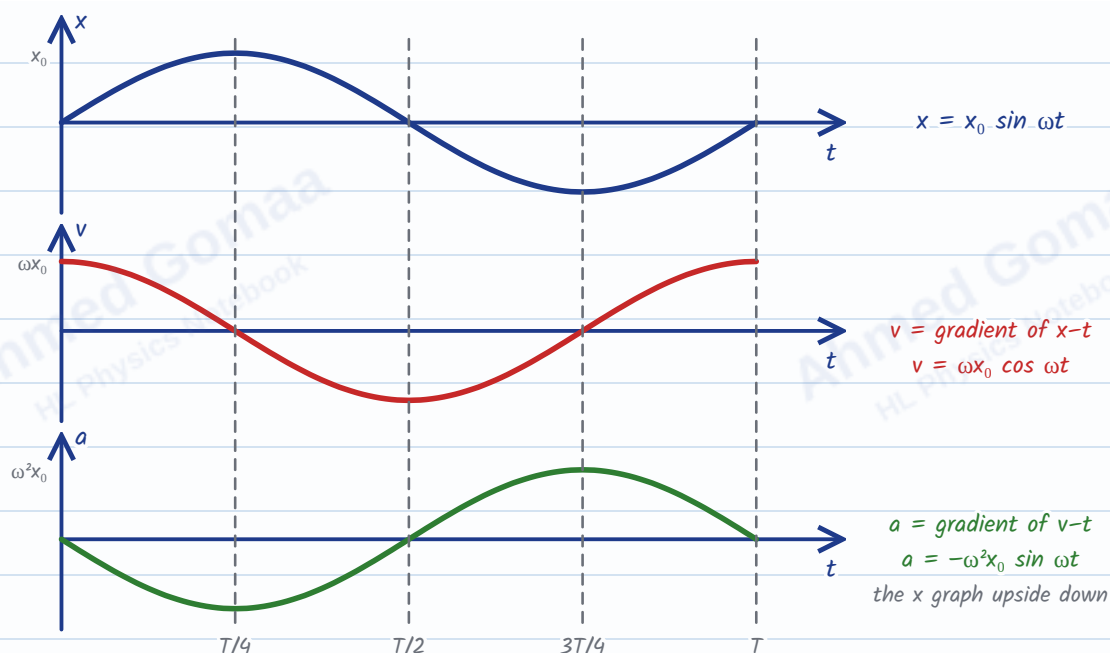
Taking ω as the gradient. The gradient is $-\omega^2$: take the square root. And drop the minus sign only after you have used it ($\omega^2 = -\text{gradient}$ is positive).



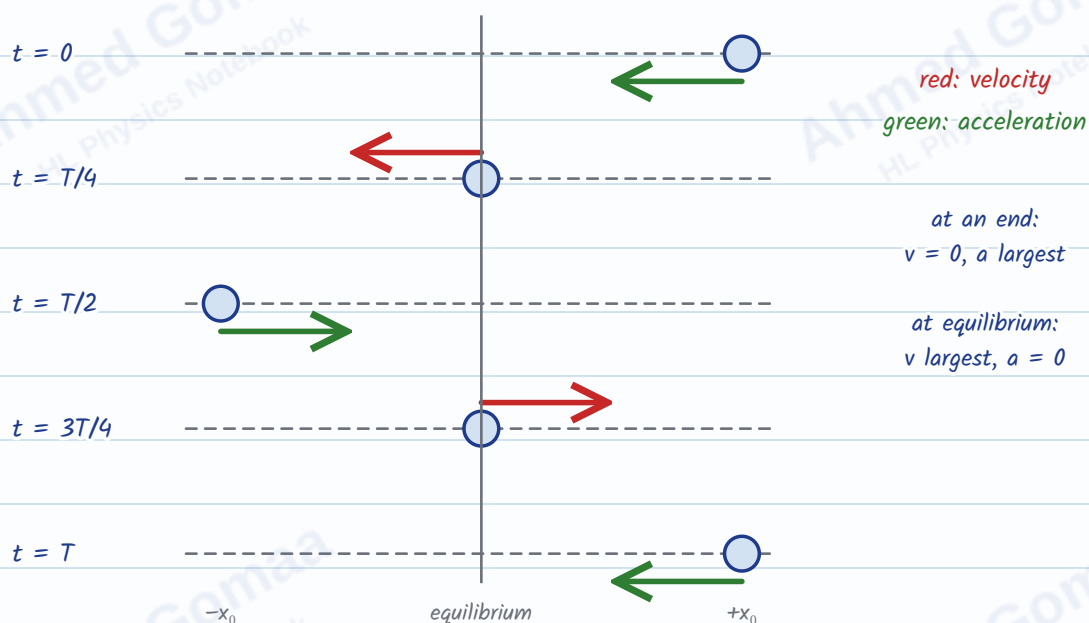
Only A is SHM. A force-displacement graph ($F = ma$) has exactly the same shape: for SHM it is a straight line through the origin with a negative gradient, $-m\omega^2$ (for a spring, $-k$).

Displacement, velocity and acceleration graphs

Velocity is the gradient of the displacement-time graph, and acceleration is the gradient of the velocity-time graph. Where x is at a peak, the $x-t$ graph is flat, so $v = 0$. Where x crosses zero, the $x-t$ graph is steepest, so v is largest. And $a = -\omega^2 x$ means the $a-t$ graph is the $x-t$ graph upside down.



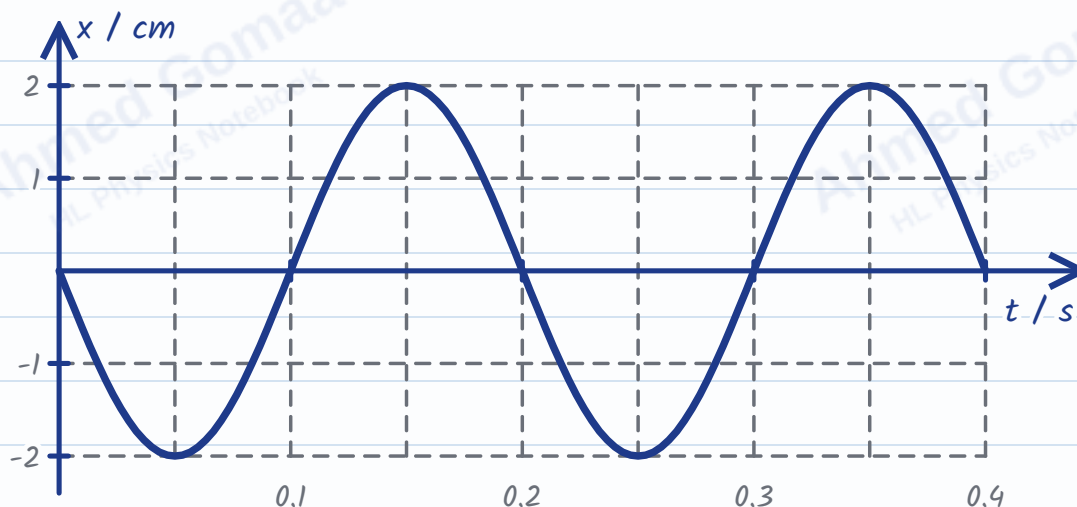
The three graphs, one quarter of a period apart. The velocity graph reaches each peak a quarter of a period before the displacement graph.



One cycle, released from $+x_0$. Velocity is largest in the middle; acceleration is largest at the ends.

Position	Speed	Acceleration	Kinetic energy	Potential energy
equilibrium ($x = 0$)	maximum	zero	maximum	zero

an end ($x = \pm x_0$) zero maximum, towards the centre zero maximum



Graph for worked example 7.

Worked example 7

The graph shows the displacement x of a particle against time t . (a) Find the period and the angular frequency. (b) Find the maximum speed and the maximum acceleration. (c) Give a time when the velocity is zero, a time when the velocity is positive and largest, and a time when the acceleration is positive and largest.

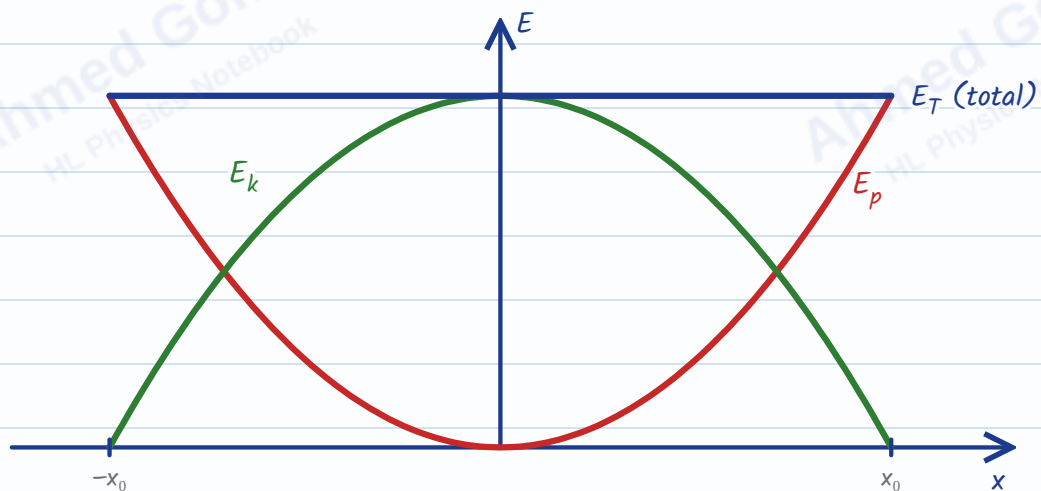
1. (a) One full cycle takes $T = 0.20 \text{ s}$. $\omega = 2\pi/T = 2\pi/0.20 = 31.4 \text{ rad s}^{-1}$.
2. (b) $x_0 = 2.0 \text{ cm} = 0.020 \text{ m}$. $v_{\max} = \omega x_0 = 31.4 \times 0.020 = 0.63 \text{ m s}^{-1}$. $a_{\max} = \omega^2 x_0 = 31.4^2 \times 0.020 = 20 \text{ m s}^{-2}$.
3. (c) Velocity zero: at a peak or trough, e.g. $t = 0.05 \text{ s}$. Velocity positive and largest: where the graph rises most steeply, e.g. $t = 0.10 \text{ s}$. Acceleration positive and largest: where x is most negative, e.g. $t = 0.05 \text{ s}$.

Same direction or opposite?

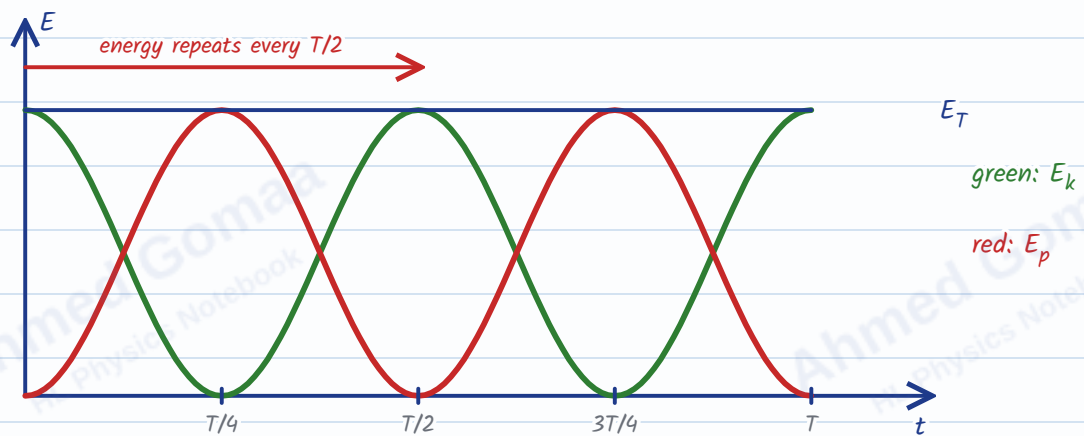
Velocity and acceleration are in the **same** direction while the object moves **towards** equilibrium (it speeds up), and in **opposite** directions while it moves away (it slows down).

Energy in SHM

As the object oscillates, energy is transferred back and forth between kinetic energy and potential energy (elastic potential energy for a spring, gravitational for a pendulum). With no friction or air resistance, the total energy stays constant.



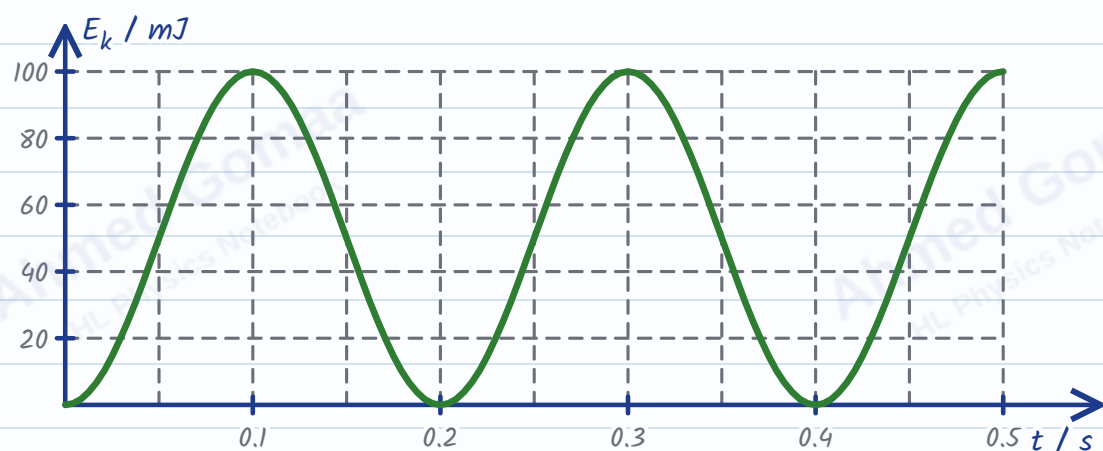
Energy against displacement: two parabolas that always add up to the same total.



Energy against time, starting at equilibrium. Each energy repeats every $T/2$: the kinetic energy is largest each time the object passes the centre, which happens twice per period.

“Describe the energy changes during one oscillation” (2–3 marks)

- ✓ At the ends ($x = \pm x_0$): all the energy is **potential**, the kinetic energy is zero.
- ✓ At the equilibrium position: all the energy is **kinetic** (maximum speed), the potential energy is zero (smallest).
- ✓ In between, energy is transferred from kinetic to potential and back; the **total energy is constant** (no friction).



Graph for worked example 8.

Worked example 8

The graph shows the kinetic energy of a particle in SHM against time. Find the period of the oscillation.

1. At $t = 0$ the kinetic energy is zero: the particle is at one end.
2. At $t = 0.20$ s the kinetic energy is zero again: the particle is at the other end, half a cycle later.
3. It is back at the start at $t = 0.40$ s, so $T = 0.40$ s.

✗ Mistake

Reading the period of an energy graph as the period of the motion. The energy repeats every $T/2$, so the period of the motion is **twice** the period of the energy graph.

HL The equations, by differentiating

We do not need to learn a new formula for each quantity. Start from the displacement, and differentiate: velocity is the rate of change of displacement, and acceleration is the rate of change of velocity.

From x to v to a

1. Displacement (clock started at the equilibrium position, moving in the + direction): $x = x_0 \sin \omega t$.
2. Velocity = dx/dt . The derivative of $\sin \omega t$ is $\omega \cos \omega t$: $v = \omega x_0 \cos \omega t$.
3. Acceleration = dv/dt . The derivative of $\cos \omega t$ is $-\omega \sin \omega t$: $a = -\omega^2 x_0 \sin \omega t$.
4. But $x_0 \sin \omega t$ is x , so $a = -\omega^2 x$. The defining equation comes back: these equations really are SHM.

5. The largest value of $\sin$ or $\cos$ is 1, so: $v_{\max} = \omega x_0$ and $a_{\max} = \omega^2 x_0$.

$$x = x_0 \sin \omega t \rightarrow v = \omega x_0 \cos \omega t \rightarrow a = -\omega^2 x_0 \sin \omega t = -\omega^2 x$$

$$v_{\max} = \omega x_0 \cdot a_{\max} = \omega^2 x_0 \cdot \text{calculator in radians}$$

Where does the clock start?

• Starts at the **equilibrium position**: $x = x_0 \sin \omega t$, so $v = \omega x_0 \cos \omega t$.

• Starts at an **end** (released from rest at x_0): $x = x_0 \cos \omega t$, so $v = -\omega x_0 \sin \omega t$ and $a = -\omega^2 x_0 \cos \omega t$.

Either way, differentiate the x equation you have. You never need a separate formula.

Worked example 9

The displacement of a particle, in metres, is $x = 0.030 \sin(12t)$, with t in seconds. (a) Find the period and the frequency. (b) Write the equations for v and a , and find $v_{\max}$ and $a_{\max}$. (c) Find x , v and a at $t = 0.10$ s.

1. (a) $\omega = 12 \text{ rad s}^{-1}$. $T = 2\pi/12 = 0.52 \text{ s}$, $f = 12/(2\pi) = 1.9 \text{ Hz}$.

2. (b) Differentiate: $v = 0.030 \times 12 \cos(12t) = 0.36 \cos(12t)$, and $a = -0.36 \times 12 \sin(12t) = -4.32 \sin(12t)$. So $v_{\max} = 0.36 \text{ m s}^{-1}$ and $a_{\max} = 4.3 \text{ m s}^{-2}$.

3. (c) $12t = 1.2 \text{ rad (radians!)}$. $x = 0.030 \sin 1.2 = 0.028 \text{ m}$; $v = 0.36 \cos 1.2 = 0.13 \text{ m s}^{-1}$; $a = -\omega^2 x = -144 \times 0.028 = -4.0 \text{ m s}^{-2}$.

Worked example 10

A particle oscillates in SHM with amplitude 5.0 mm and frequency 0.40 Hz. At $t = 0$ it is at rest at $x = +5.0$ mm. (a) Write the equations for x and v . (b) Find the velocity the first time the displacement is 2.5 mm.

1. (a) $\omega = 2\pi f = 2\pi \times 0.40 = 2.51 \text{ rad s}^{-1}$. It starts at an end, so $x = 5.0 \cos(2.51t)$ (mm).
2. Differentiate: $v = -12.6 \sin(2.51t)$ (mm s⁻¹).
3. (b) $2.5 = 5.0 \cos(2.51t)$, so $\cos(2.51t) = 0.5$ and $2.51t = \pi/3$ (first time), $t = 0.42 \text{ s}$.
4. $v = -12.6 \sin(\pi/3) = -10.9 \text{ mm s}^{-1}$ (negative: moving back towards the centre).

Worked example 11

In an SHM the maximum speed is 3.0 m s^{-1} and the maximum acceleration is 24 m s^{-2} . Find the period and the amplitude.

1. $a_{\max}/v_{\max} = \omega^2 x_0 / (\omega x_0) = \omega$, so $\omega = 24/3.0 = 8.0 \text{ rad s}^{-1}$.
2. $T = 2\pi/8.0 = 0.79 \text{ s}$.
3. $x_0 = v_{\max}/\omega = 3.0/8.0 = 0.38 \text{ m}$.

✗ Mistake

Leaving the calculator in degrees. ωt is an angle in **radians**: $\sin(1.2)$ in degrees gives 0.021 instead of 0.932.

HL Speed at any displacement

Often a question gives a displacement, not a time. Then use the data-booklet equation that links v and x directly. It comes from the two equations above: $\sin^2 + \cos^2 = 1$, so $(x/x_0)^2 + (v/\omega x_0)^2 = 1$.

$$v = \pm \omega \sqrt{(x_0^2 - x^2)} \text{ (data booklet)}$$

$\pm$ because at the same x the object can be moving either way · at $x = 0$, $v = \omega x_0$ (maximum) · at $x = \pm x_0$, $v = 0$

Worked example 12

A point oscillates in SHM with amplitude 6.0 cm and period 1.5 s. Find its speed when the displacement is 3.0 cm.

1. $\omega = 2\pi/1.5 = 4.19 \text{ rad s}^{-1}$.

2. $v = \omega \sqrt{(x_0^2 - x^2)} = 4.19 \times \sqrt{(6.0^2 - 3.0^2)} = 4.19 \times \sqrt{27} = 21.8 \text{ cm s}^{-1}$.

3. Note: half the amplitude gives $\sqrt{3}/2 = 0.87$ of the maximum speed, not half of it.

HL Phase angle: reading

the data booklet equation

So far we started the clock at a special moment: at the centre ($x = x_0 \sin \omega t$) or at an end ($x = x_0 \cos \omega t$). The data booklet gives one equation that works **whatever moment** the clock starts:

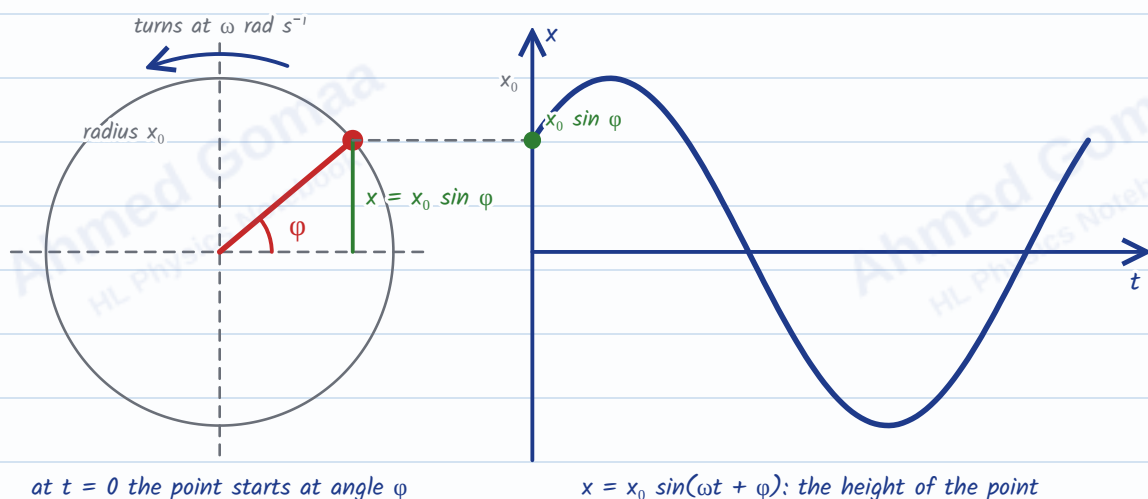
$$x = x_0 \sin(\omega t + \phi) \cdot v = \omega x_0 \cos(\omega t + \phi) \text{ (data booklet)}$$

Part	Name	What it tells you
x_0	amplitude	how far the object goes from the centre
ω	angular frequency	how fast it oscillates: $T = 2\pi/\omega$

$(\omega t + \varphi)$ phase, in rad where the object is in its cycle at time t ; one full cycle = 2π rad

φ phase angle the phase at $t = 0$: where the object is, and which way it moves, when the clock starts

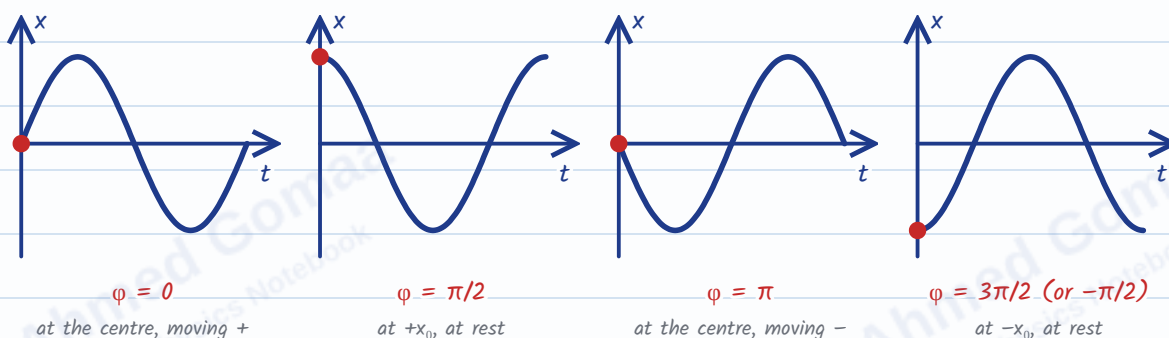
Think of a point going round a circle of radius x_0 at ω radians per second. Its height above the centre line moves up and down in SHM: $x = x_0 \sin(\text{angle})$. If the point is already at angle φ when the clock starts, the angle at time t is $\omega t + \varphi$. That is all φ is: **the starting angle**. It does not change the amplitude, the period or the energy.



The phase angle is the starting angle of the point on the circle. Here $\varphi = 40^\circ$, so the graph starts at $x_0 \sin 40^\circ$, not at 0.

It is the same as our method

Differentiate the booklet equation exactly as before: $x = x_0 \sin(\omega t + \varphi)$
 $\rightarrow v = \omega x_0 \cos(\omega t + \varphi) \rightarrow a = -\omega^2 x_0 \sin(\omega t + \varphi) = -\omega^2 x$. The ω comes out in front each time; φ just rides along inside the bracket. With $\varphi = 0$ you get our $x = x_0 \sin \omega t$, and with $\varphi = \pi/2$ you get $x = x_0 \cos \omega t$, because $\sin(\theta + \pi/2) = \cos \theta$.



The four starting points that come up most. Each graph is the same oscillation, only started at a different moment.

How to find φ

1. Put $t = 0$ in both equations: $x(0) = x_0 \sin \varphi$ and $v(0) = \omega x_0 \cos \varphi$.
2. Use $x(0)$ to get $\sin \varphi$. Between 0 and 2π there are **two** angles with the same sine.
3. Use the direction of motion to choose: moving in the + direction at $t = 0$ means $\cos \varphi > 0$; moving in the - direction means $\cos \varphi < 0$.

Worked example 13

An object oscillates in SHM with amplitude 0.20 m. At $t = 0$ its displacement is +0.10 m and it is moving **towards** the equilibrium position. Find φ .

1. $x(0) = x_0 \sin \varphi$: $0.10 = 0.20 \sin \varphi$, so $\sin \varphi = 0.5$, and $\varphi = \pi/6$ or $5\pi/6$.
2. It is at +0.10 m and moving towards the centre, so it moves in the - direction: $v(0) < 0$, so $\cos \varphi < 0$.
3. $\cos(\pi/6) = +0.87$ and $\cos(5\pi/6) = -0.87$, so $\varphi = 5\pi/6$ (2.6 rad).
The calculator gives only $\pi/6$, which is the answer for moving **away** from the centre.

Worked example 14

The displacement of a loudspeaker cone, in metres, is $x = 0.300 \sin(4\pi t + \pi/2)$. (a) Find the frequency. (b) Where is the cone at $t = 0$? (c) Find the maximum speed and the maximum acceleration. (d) Find x and v at $t = 0.10$ s.

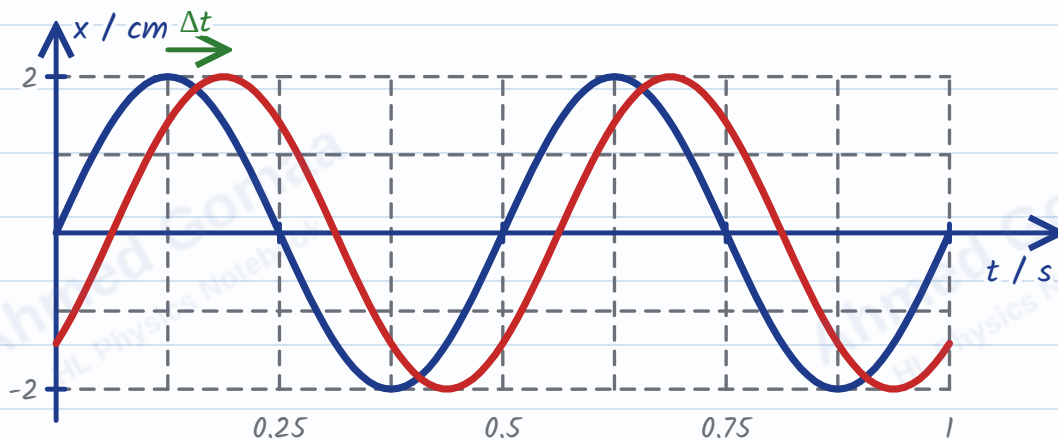
1. (a) $\omega = 4\pi \text{ rad s}^{-1}$, so $f = \omega/2\pi = 2.0 \text{ Hz}$ ($T = 0.50 \text{ s}$).
2. (b) $\phi = \pi/2$: $x(0) = 0.300 \sin(\pi/2) = 0.300 \text{ m}$, and $v(0) = \omega x_0 \cos(\pi/2) = 0$. The cone starts at $+x_0$, at rest.
3. (c) $v_{\max} = \omega x_0 = 4\pi \times 0.300 = 3.8 \text{ m s}^{-1}$; $a_{\max} = \omega^2 x_0 = (4\pi)^2 \times 0.300 = 47 \text{ m s}^{-2}$.
4. (d) Phase $= 4\pi \times 0.10 + \pi/2 = 2.83 \text{ rad}$. $x = 0.300 \sin(2.83) = 0.093 \text{ m}$; $v = 3.77 \cos(2.83) = -3.6 \text{ m s}^{-1}$ (moving back towards the centre). Check with $x = x_0 \cos \omega t$: $0.300 \cos(1.26) = 0.093 \text{ m}$, the same.

Reading ϕ in an exam

- ✓ $\phi = 0 \rightarrow$ starts at the centre, moving $+$. $\phi = \pi/2 \rightarrow$ starts at $+x_0$. $\phi = \pi \rightarrow$ centre, moving $-$. $\phi = 3\pi/2 \rightarrow$ starts at $-x_0$.
- ✓ ϕ must be in radians, and the calculator too.
- ✓ ω is the number in front of t , never $\omega + \phi$.

Phase difference

Two oscillations with the same period can be at different points of their cycles. The **phase difference** says how far apart they are, as an angle.



Same oscillation, different starting times. The phase difference comes from the time shift Δt .

Phase difference: $\Delta\phi = (\Delta t/T) \times 2\pi$ (rad), or the difference of the two phase angles

one whole period = 2π rad (in phase again) · half a period = π rad (anti-phase) · a quarter = $\pi/2$

Worked example 15

In the graph above, find the phase difference between the two oscillations.

1. The period is $T = 0.50$ s, and the red peak comes $\Delta t = 0.0625$ s after the blue peak.
2. $\Delta\phi = (0.0625/0.50) \times 2\pi = \pi/4$ rad (0.79 rad).

Phase differences inside one oscillation

From the graphs of x , v and a : the velocity is $\pi/2$ ahead of the displacement (it peaks a quarter period earlier), and the acceleration is π out of phase with the displacement ($a = -\omega^2 x$: anti-phase).

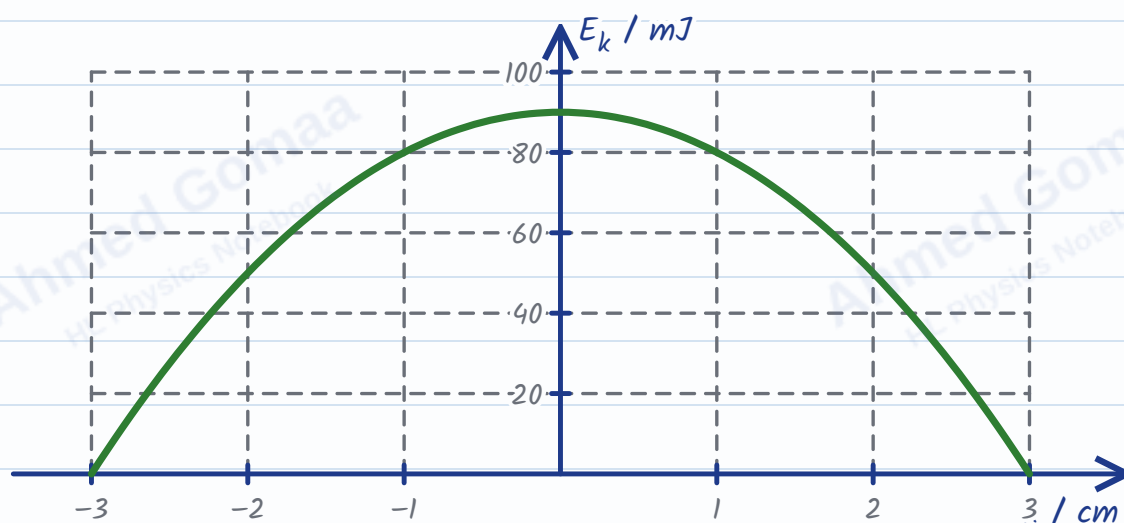
HL Energy equations

The kinetic energy is $\frac{1}{2}mv^2$. Put $v_{\max} = \omega x_0$ in it and you get the total energy; put $v = \omega\sqrt{(x_0^2 - x^2)}$ in it and you get the kinetic energy at any x . The rest is potential energy.

$$E_T = \frac{1}{2}m\omega^2 x_0^2 \cdot E_p = \frac{1}{2}m\omega^2 x^2 \text{ (data booklet)}$$

$$E_k = E_T - E_p = \frac{1}{2}m\omega^2 (x_0^2 - x^2)$$

the total energy $\propto x_0^2$: double the amplitude, four times the energy



Graph for worked example 16.

Worked example 16

The graph shows the kinetic energy of a 0.50 kg mass in SHM against its displacement. Find (a) the total energy, (b) the maximum speed, (c) the period, (d) the potential and kinetic energies at $x = 1.5$ cm.

1. (a) At $x = 0$ all the energy is kinetic: $E_T = 90 \text{ mJ}$.
2. (b) $\frac{1}{2}mv_{\max}^2 = 0.090 \text{ J}$, so $v_{\max} = \sqrt{(2 \times 0.090/0.50)} = 0.60 \text{ m s}^{-1}$.
3. (c) The amplitude is the end of the curve, $x_0 = 3.0 \text{ cm}$. $\omega = v_{\max}/x_0 = 0.60/0.030 = 20 \text{ rad s}^{-1}$; $T = 2\pi/20 = 0.31 \text{ s}$.
4. (d) $E_p \propto x^2$: at half the amplitude $E_p = (1/2)^2 \times 90 = 22.5 \text{ mJ}$
(check: $\frac{1}{2} \times 0.50 \times 20^2 \times 0.015^2 = 0.0225 \text{ J}$). $E_k = 90 - 22.5 = 67.5 \text{ mJ}$.

Worked example 17

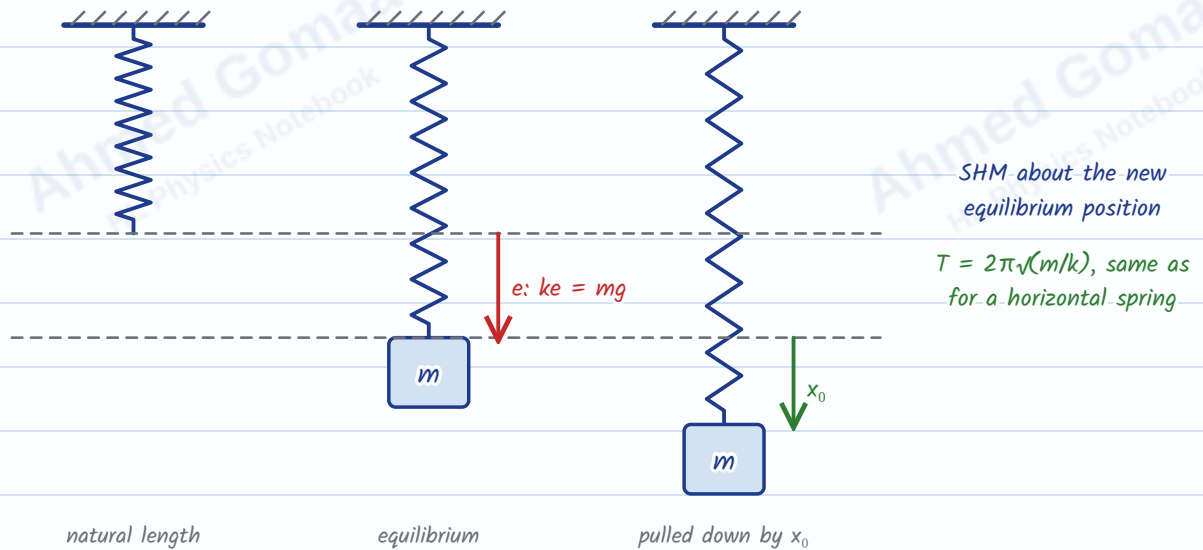
A 0.50 kg body moves in SHM with amplitude 0.20 m and period $\pi/5$ s. (a) Find its total energy. (b) At what displacement are its kinetic and potential energies equal?

1. (a) $\omega = 2\pi/T = 2\pi/(\pi/5) = 10 \text{ rad s}^{-1}$. $E_T = \frac{1}{2}m\omega^2x_0^2 = \frac{1}{2} \times 0.50 \times 10^2 \times 0.20^2 = 1.0 \text{ J}$.
2. (b) $E_p = E_k = \frac{1}{2}E_T$, so $\frac{1}{2}m\omega^2x^2 = \frac{1}{2} \times \frac{1}{2}m\omega^2x_0^2$, giving $x^2 = x_0^2/2$ and $x = x_0/\sqrt{2} = \pm 0.14 \text{ m}$ (not 0.10 m).

HL The vertical spring

Hang the mass on a vertical spring. It now rests lower, at the point where the spring force balances the weight: $ke = mg$, where e is the extension. This is the new equilibrium position. Measured from **there**, the extra force

for a further displacement x is still $-kx$, so the motion is SHM with the same period $T = 2\pi\sqrt{m/k}$.



Gravity only moves the centre of the oscillation down. It does not change the period.

Important note

For a vertical spring, $\frac{1}{2}m\omega^2x^2$ is the **oscillation** energy measured from the equilibrium position. It is **not** the elastic energy stored in the spring. The spring stores $\frac{1}{2}k(\text{extension})^2$, and the extension is $e + x$, not x . Some energy also goes into gravitational potential energy: as the mass rises, kinetic energy and elastic energy are transferred to gravitational energy.

Worked example 18

A 0.20 kg mass hangs at rest on a vertical spring with $k = 8.0 \text{ N m}^{-1}$ ($g = 9.8 \text{ m s}^{-2}$). (a) Find the extension and the elastic energy stored in the spring. (b) The mass is pulled down 0.10 m and released. Find the period and the maximum speed.

1. (a) $ke = mg$, so $e = 0.20 \times 9.8/8.0 = 0.245 \text{ m}$. Elastic energy = $\frac{1}{2}ke^2 = \frac{1}{2} \times 8.0 \times 0.245^2 = 0.24 \text{ J}$.
2. (b) $T = 2\pi\sqrt{m/k} = 2\pi\sqrt{0.20/8.0} = 0.99 \text{ s}$. $\omega = \sqrt{k/m} = 6.32 \text{ rad s}^{-1}$; $v_{\max} = \omega x_0 = 6.32 \times 0.10 = 0.63 \text{ m s}^{-1}$.
3. Note: the oscillation energy is $\frac{1}{2}m\omega^2 x_0^2 = 0.040 \text{ J}$, but the elastic energy at the lowest point is $\frac{1}{2} \times 8.0 \times 0.345^2 = 0.48 \text{ J}$. They are different quantities.

Writing the explanations

These are the explanations C.I asks most often, with the words the markschemes look for.

"State the conditions for simple harmonic motion" (2 marks)

- ✓ The acceleration / restoring force is **proportional to the displacement** (from equilibrium).
- ✓ It is **directed towards the equilibrium position** / opposite to the displacement.

The equation $a = -\omega^2 x$ alone scores 0.

“Explain, using the graph, why the motion is SHM” (2 marks)

For an $a-x$ (or $F-x$) graph:

- ✓ straight line **through the origin**, so $a \propto x$;
- ✓ **negative gradient**, so a is opposite to x .

“Determine the period from the graph” (2–3 marks)

- ✓ gradient = $-\omega^2$ (show the numbers you read);
- ✓ $\omega = \sqrt{(-\text{gradient})}$;
- ✓ $T = 2\pi/\omega$ (or $f = \omega/2\pi$).

“Describe the energy transfers during one oscillation” (2 marks)

- ✓ Kinetic energy is largest at the equilibrium position and zero at the ends; potential energy is the other way round.
- ✓ Energy is transferred between kinetic and potential, and the total stays constant (no resistive forces). For a vertical spring, name gravitational potential energy too.

“State and explain the direction of motion at time t ” (2 marks)

- ✓ Give the direction in words: “up”, “to the left”. “Towards the equilibrium position” **alone** is not accepted.
- ✓ Explain with the graph: e.g. the elastic energy is decreasing, so the extension of the spring is decreasing, so the mass moves up.

"Show that the period of a pendulum does not depend on the mass" (2 marks)

✓ The restoring force is $mg \sin \theta$ and the inertia is m , so $a = -g \sin \theta \approx -(g/l)x$: the mass cancels.

✓ So $\omega^2 = g/l$ and $T = 2\pi\sqrt{l/g}$, which has no m in it.

Common mistakes

✗ Ten traps in C.1

1. Writing only $a = -\omega^2 x$ for "state the conditions for SHM". Write the two conditions in words. [0 marks for the equation alone]
2. Saying "the acceleration is proportional to the displacement" and forgetting the direction (towards equilibrium / opposite to the displacement).
3. Using $\omega^2 = k/m$ for a system that is not a mass on a spring. Use ω^2 from the data or the graph.
4. Taking ω as the gradient of the a - x graph. The gradient is $-\omega^2$; take the square root.
5. Measuring the amplitude from end to end (that is $2x_0$), or reading the period from one zero crossing to the next (that is $T/2$).
6. Reading the period of an energy-time graph as the period of the motion. Energy repeats every $T/2$.
7. HL: calculator in degrees in $x = x_0 \sin \omega t$. Always radians.
8. HL: using $x = x_0 \sin \omega t$ when the object starts at an end (then $\phi = \pi/2$; $x = x_0 \cos \omega t$), or taking the calculator's ϕ from $\sin \phi$ without checking the direction of motion (two angles have the same sine).
9. HL: thinking $E_k = E_p$ at $x = x_0/2$. It happens at $x = x_0/\sqrt{2}$ (at $x_0/2$, E_p is only a quarter of the total).
10. HL: for a vertical spring, using $\frac{1}{2}m\omega^2 x^2$ as the elastic energy in the spring. The spring stores $\frac{1}{2}k(\text{extension})^2$, and the extension at equilibrium is not zero.

Practise next

You have read the notes. Now practise, in this order:

1. **Levelled practice**: first a ★ at Level 2 in every topic (the goal), then Levels 3 and 4 for a 6 or 7. Hints and worked answers help you as you go.
2. **IB-format practice** ([Paper 1A](#) · [Paper 2](#)): timed, one attempt, and your report goes to your teacher. Do these when the practice goal is reached.

© 2026 Ahmed Gomaa · HL Physics Notebook · Original material. All rights reserved. Free for personal study. Teachers: please [ask](#) before sharing or printing for a class.

Not affiliated with or endorsed by the International Baccalaureate Organization.

© 2026 Ahmed Gomaa · All rights reserved · Free for personal study · Not affiliated with the IBO